Problem Solving: 11

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11.

Recall the exploding combination lock problem 4 from last week, which said

> A safe is protected by a combination lock. The code is a one-digit number. The lock has two properties: if you enter a wrong code on the keypad then the code increases by 1 modulo 10; if you enter the same number for the second time in a row and it proves to be wrong, the whole safe explodes. Is it possible to reliably unlock the safe?

Now answer: is it possible to unlock the safe if once you try a number, you can never try it again?

Solution

The first step is to identify the restrictions on the numbers we can guess. Let $g \in \mathbb{Z}_n$ be the number of the guess. Let $x_m \in \mathbb{Z}_n$ be the value of the mth guess where $g = m$. Let $k \in \mathbb{Z}_n$.

1. Guessing x_m eliminates all guesses $x_m + k \mod n$ on guess $g = m + k$. This is because the code increases by 1 for each guess, so guessing $x_m + k$ will result in the elimination of a number we have already eliminated with guess $g = m$. Because we only have n guesses and n possible codes, we cannot afford to eliminate the same number twice.

2. We can only guess 1 number per guess. So guessing x_m eliminates $x_m - k$ for any $k \neq m$ on $g = m$.

3. We cannot guess x_m more than once. Guessing x_m eliminates us from guessing x_m on $g = k$ where $k \neq m$.

Using these restrictions, we can create an n x n grid. Each column represents a guess g, while each row represents x_g. The restrictions translate to the grid,

1. No two guesses x_m and x_k on the grid can share the same minor diagonal. (Diagonals wrap around the grid $\mod n$.)

2. No two guesses x_m and x_m can share the same column.

3. No two guesses x_m and x_k can share the same row.

The question becomes: Can we place n objects in the n x n toroidal grid such that no two share the same column, row, or minor diagonal.

From 2 and 3, we can say that x_m is a permutation of $\{0, 1, ..., n-1\}$. In order to make sure that no two x_m share the same minor diagonal, suppose a new permutation exists,

$$y_m = x_m + m$$

From the definition of a permutation, we know,

$$\sum_{m=0}^{n-1} y_m = \sum_{m=0}^{n-1} m = \sum_{m=0}^{n-1} x_m + m = 2\sum_{m=0}^{n-1} m = n(n-1) = 0 \mod n$$

Because $\sum_{m=0}^{n-1} m = \sum_{m=0}^{n-1} x_m$. So,

$$\sum_{m=0}^{n-1} m = \frac{n(n-1)}{2} = 0 \mod n$$

This means that $\frac{n-1}{2}$ must be an integer, meaning n cannot be even. So for a safe with 10 possible 1 digit combinations, we cannot reliably unlock the safe without guessing a number more than once.

Note: This does not imply there is a solution for odd n, because y_m may not exist for odd n.

I used some help on the proof from the N-Queens Problem activity book by Alicia Sanchez. This problem is the same as the Toroidal N-Semi-queens problem.